Covariance propagation for such systems is treated elsewhere,[4,5] and in general requires an additional matrix differential equation because of the time delay.

Adjoint Solutions

In terminal control (e.g., missile intercept) problems, where chiefly the final values of the states are of interest, the same basic concept can be applied in adjoint analysis.[6] It is evident from Fig. 2 that the impulse response of the augmented system is the same as the response of the original system to the specified disturbance. Hence the adjoint approach, which basically generates impulse responses in a particular state at the final time, to disturbances at all previous times, can be applied in cases where those disturbances are not impulses, but have some other (known) form. By appropriate squaring, weighting, and integrating operations on the adjoint solutions, one can thus also generate the mean-square final output when the disturbance starting time is random.[6]

In the case of systems containing time delays, adjoint studies can be carried out in a straightforward (but rather tedious) manner. In general, the adjoint version of a system may be obtained from the system block diagram simply by reversing all flow directions and interchanging summers and branch points.[1,6] It can be shown that this rule still holds true when the system includes a pure time delay (i.e., the adjoint of a delay is a delay). Thus, the generation of the adjoint solutions requires state propagation in a system containing a delay. This can be accomplished by storing the input to the delay in a pushdown list of length T, where T is the delay time.

Discussion

The shaping-filter technique described here is of great usefulness in the analysis of linear systems, where it enables the analyst, by appending a few state variables to the system, to generate second-order statistical properties of the output when the input is of known form but has a random starting time. The method can be applied directly to multidimensional systems, where it can be used in propagating the complete covariance matrix of the system states or in generating influence functions via adjoint solutions.

The system under consideration may include an estimator, and the estimation errors may constitute system states. Thus the method can be used to analyze estimator performance and to investigate the effects of changes in estimator parameters or configuration.

The method can be extended to mildly nonlinear systems by using it in conjunction with random input describing functions to allow covariance propagation.[2] Some examples of this are given in Refs. 4 and 7.

In the application of this approach to covariance propagation, as in any approach based on mean-square statistics, it should be realized that the details of the system transient response to abrupt disturbances may be more important than ensemble mean-square values. For example, an abrupt target maneuver may cause a tracking radar to lose track. In such problems, mean-square statistics should not be the only design criterion. When the mean-square criterion is applicable, however, the shaping-filter approach provides a powerful tool for system analysis and design.

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A Comparison of Different Forms of Dirigible Equations of Motion

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Introduction

SEVERAL formulations of airship equations of motion are currently in use in the lighter-than-air (LTA) industry. A significant difference in the equations comes about depending on whether the apparent masses and apparent inertias are treated as added masses or added inertias, or whether they are treated as aerodynamic acceleration and moment derivatives.

One way of writing the dirigible equations of motion is to add the additional mass and additional inertia to the physical mass and physical inertias of the structure, equipment, and lifting gas. For convenience we will refer to this form of the equations as the "additional mass and inertia" form. An example from this set of equations[1] is the side-force perturbation equation (in body axes):

$$\left(m_s+m_g+m_{a_2}-\frac{\rho S_R\bar{c}}{4}C_{Y_{\dot\beta}}\right)\ddot{y}_b-\frac{\rho V_\infty S_R}{2}C_{Y_\beta}\dot{y}_b+\left[(m_s+m_g+m_{a_2})V_\infty-\frac{\rho V_\infty S_R\bar{c}}{4}C_{Y_r}\right]\dot\psi_b=T\delta_Y \qquad (1)$$

where m_s, m_g and m_{a2} are the structural, lifting gas, and apparent masses, respectively; see Ref. 1 for the balance of nomenclature.

Clark[2] has derived fully coupled, six-degree-of-freedom equations of motion, linearized them, and separated them into trim equations and a set of perturbation equations in body axes. Clark's side-force equation is, with some simplifying assumptions to permit comparison with Eq. (1),

$$\left(m_s+m_g+m_{a_2}-\frac{\rho S_R\bar{c}}{4}C_{Y_{\dot\beta}}\right)\ddot{y}_b-\frac{\rho V_\infty S_R}{2}C_{Y_\beta}\dot{y}_b+\left[(m_s+m_g)V_\infty-\frac{\rho V_\infty S_R\bar{c}}{4}C_{Y_r}\right]\dot\psi_b=T\delta_y \qquad (2)$$

For convenience we will refer to this form of the equation as the "aero-acceleration" form.

The centrifugal-force term of Eq. (1) includes the apparent additional mass m_{a2}. But since m_{a2} is really an aerodynamic term, it should not be included with the physical masses of the lifting gas and the structure (m_g and m_s, respectively). None of the dimensional derivatives in Table 1 (such as N_r, N_β, L_r,

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Table 1 Linear perturbation equations – aero-acceleration form

Yaw/roll equations[a]	
Yawing moment:	$\left[(1-N_{\dot r})S^2 - N_r S\right]\Delta\psi - \left(\frac{I_{xz}S^2}{I_{zz}}\right)\Delta\phi - (N_\beta)\Delta\beta = (N_\delta)\Delta\delta_Y$
Rolling moment:	$\left(\frac{-I_{xz}S^2}{I_{xx}} - L_r S\right)\Delta\psi + \left[(1-L_{\dot p})S^2 - L_p S - L_\phi\right]\Delta\phi - (L_\beta)\Delta\beta = (L_\delta)\Delta\delta_Y$
Side force:	$[(V_\infty - Y_r)S]\Delta\psi + (0.)\Delta\phi + \left[V_\infty(1.-Y_{\dot v})S - Y_\beta\right]\Delta_\beta = (Y_\delta)\Delta\delta_Y$
Pitch equations	
Normal force:	$\left[V_\infty(1-Z_{\dot z})S - Z_\alpha\right]\Delta\alpha - [(V_\infty + Z_q)S]\Delta\theta = (Z_\delta)\Delta\delta_P$
Pitching moment:	$-(M_\alpha)\Delta\alpha + \left[(1-M_{\dot q})S^2 - M_q \cdot S - M_\theta\right]\Delta\theta = (M_\delta)\Delta\delta_P$

[a] See Ref. 3., pp. 68-71, inclusive, for nomenclature.

L_p, Y_r, Y_β) include apparent masses or inertias. Table 1 shows the equations of motion in the aero-acceleration form[3] (except for the x-force equation, which is negligible for this problem). The encircled terms of the Table 1 equations include the apparent mass and apparent inertias as proportionality constants for linear and angular acceleration terms. Clark (Ref. 2, pp. 20, 32, and B3) describes $Z_{\dot w}$, $Y_{\dot v}$ (i.e., the $(\rho S_R \bar{c}/4)\, C_{Y_{\dot\beta}}$ term above), etc., as the "lag due to downwash effect." These effects were negligible for the Akron and are assumed negligible for the High Altitude Super Pressured Aerostat (HASPA).

No studies that resolve the differences between the sets of equations represented by Eq. (1) and (2) have been found. Nor has there been any opportunity to correlate the predicted LTA dynamic responses with inflight data. However, in lieu of LTA flight test data, the equations can be verified by turning to experience with underwater vehicles.[4,5]

Underwater Equations

The standard equations of motion[5] for submarine simulation have been used "to conduct simulation studies of submarines engaged in submerged maneuvers ranging from normal maneuvers to extreme maneuvers .... Studies involving about 25 different designs have been carried out .... A comparison program is also being carried out to determine the extent to which computer predictions, involving the use of the standard equations, agree with measured trajectories .... comparisons made to date have generally shown good agreement between predicted and measured trajectories ...." The standard submarine equations treat apparent masses and inertias in the same manner as shown in Eq. (2) and Table 1.

Lindgren,[4] using a nonlinear model based on the submarine equations, showed good comparison between the transient behavior in water and computer-simulated transients. Lindgren's equations are in body axes and in the aero-acceleration form similar to Eq. (2). The only discernible difference between his simulated response and his in-water response is that the rudder rate limit is evident as rudder deflection ramps right and left in the water, whereas in the simulation the rudder steps to its maximum deflection immediately.

These correlations indicate that the standard submarine equations adequately represent the dynamics of a fully submerged vehicle, i.e., both submarines and airships.

A Representative LTA Application

In early dynamic work on the HASPA dirigible, equations of motion based on the additional mass and inertia approach were used.[1] These equations are represented by the side-force

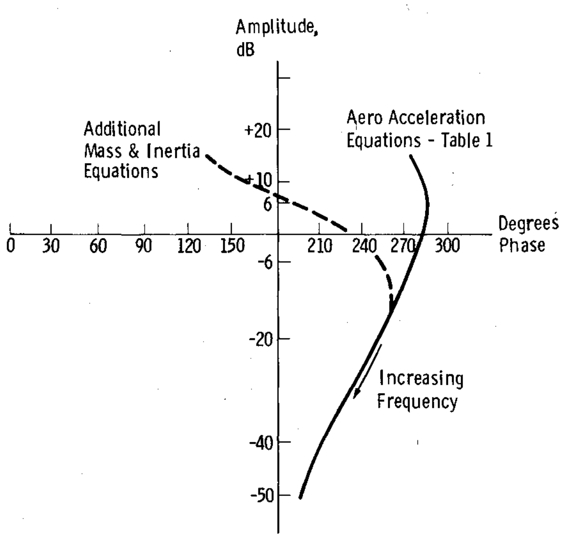

Fig. 1 Yaw roll open-loop frequency response at $V_\infty = 15$ knots with autopilot.

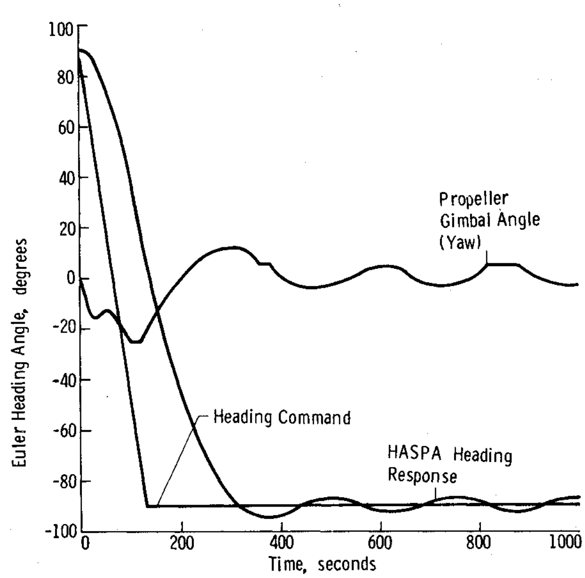

Fig. 2 HASPA nonlinear simulation response to 180-deg, heading command – additional mass and inertia equations.

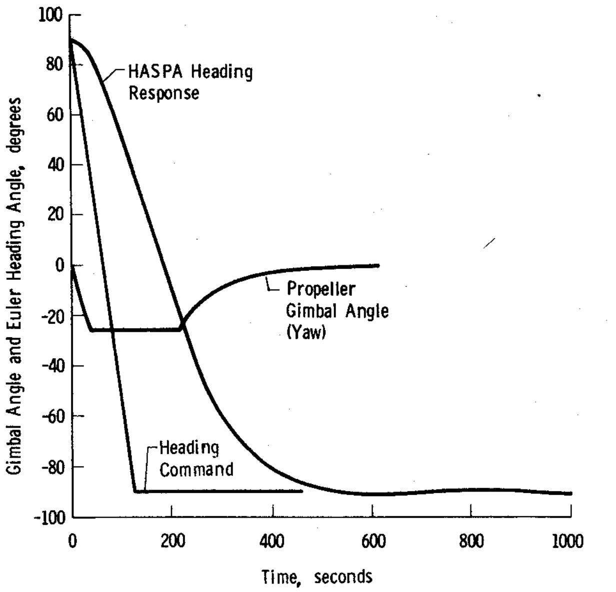

Fig. 3 HASPA nonlinear simulation response to 180-deg heading command – aero-acceleration equations (Table 1).

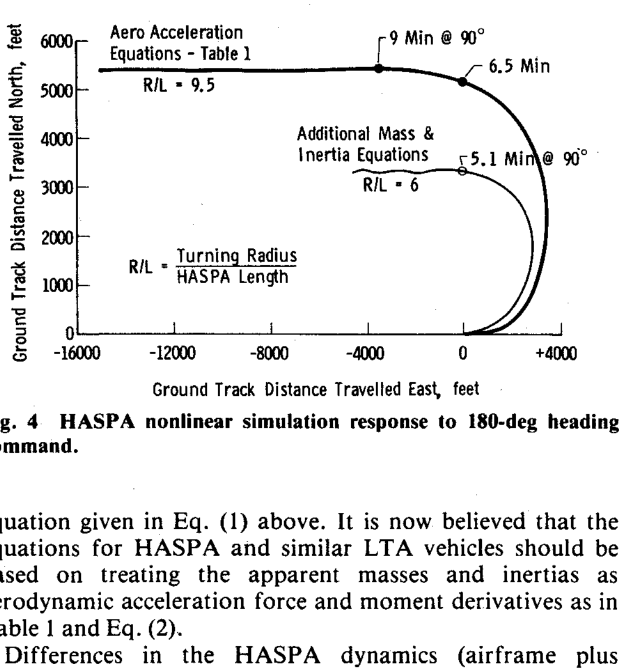

Fig. 4 HASPA nonlinear simulation response to 180-deg heading command.

equation given in Eq. (1) above. It is now believed that the equations for HASPA and similar LTA vehicles should be based on treating the apparent masses and inertias as aerodynamic acceleration force and moment derivatives as in Table 1 and Eq. (2).

Differences in the HASPA dynamics (airframe plus autopilot) as predicted by the two different sets of equations were evaluated by deriving linear transfer functions from the equations of Table 1 and from the earlier equations. From these transfer functions the Nichols plot of Fig. 1, was obtained wherein increasing stability margin is indicated by increasing distance of the frequency response locus from the point (−180 deg, 0 dB). Stability margins using the equations of Table 1 are clearly improved over the margins predicted with the additional mass and inertia form of the equations.

A 180-deg turn was simulated with both forms of the equations, including all control and aerodynamic nonlinearities. With the additional mass and inertia equations, the results of Fig. 2 were obtained. The turn command was applied at a rate of 1.4 deg/s and the resulting gimbal response [linear law: $\delta = (\psi_{ref} - \psi_b) + 75(\dot{\psi}_{ref} - \dot{\psi}_b)$] and heading angle are shown. The heading angle exhibits limit cycles of approximately ±3-deg amplitude at a frequency of 0.024 rad/s approximately 320 s after starting to turn. Figure 3 shows the same maneuver using the equations of Table 1; the turn is completed in approximately 540 s. Limit cycling is barely perceptible after the turn is complete.

Another difference between the dynamic predictions is indicated in Fig. 4, which shows the dirigible ground track in response to the same input turn command as in Figs. 2 and 3. The response from the Table 1 equations is more sluggish than the response shown by the additional mass and inertia equations ($R/L = 9.5$ and $R/L = 6.0$, respectively).

The increased stability predicted by the equations of Table 1 makes the dynamic system less sensitive to large variations which may be encountered in estimating the aerodynamic data. Also, the improved stability margins are beneficial because they minimize attitude limit cycling.

Conclusions

The form of the equations of motion for airships and the manner in which the apparent additional mass and inertias are treated can be reliably based on the proven standard equations of motion for submarines. The apparent masses and inertias are really aerodynamic force and moment acceleration derivatives and should be treated as shown in Table 1. For HASPA, this treatment of the equations predicts greater stability margins than was forecast with the equations[1] formerly used.

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General Expression for a Three-Angle Rotation Matrix

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THE solution on a digital computer of the equations of motion of a system of rigid bodies requires frequent computations of the orthogonal matrices relating various orthonormal frames introduced in the formulation of the problem. The usual method is to relate any two such frames (both assumed to be right-handed) by a sequence of, at most, three single-axis rotations, so that the complete transformation matrix R is a matrix product of the form

$$R = R_i(a) R_j(b) R_k(c) \tag{1}$$

where $R_n(x)$ denotes a rotation about axis n ($n = 1,2,$ or 3) through an angle x. A simple and efficient method for the generation of matrices of the form of Eq. (1) is thus of considerable interest.

Two fairly recent papers[1,2] have presented useful algorithms for the solution of the stated rotation matrix

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